

# Poisson-Nijenhuis structures on quiver path algebras

Claudio Bartocci and Alberto Tacchella

January 2, 2017

## Abstract

We introduce a notion of noncommutative Poisson-Nijenhuis structure on the path algebra of a quiver. In particular, we focus on the case when the Poisson bracket arises from a noncommutative symplectic form. The formalism is then applied to the study of the Calogero-Moser and Gibbons-Hernsen integrable systems. In the former case, we give a new interpretation of the bihamiltonian reduction performed in [3].

## 1 Introduction

Since Magri's seminal paper [24], the notion of bihamiltonian manifold has played a central role in the theory of integrable systems. Some of the most significant examples of bihamiltonian manifolds arise from a Poisson-Nijenhuis (PN) structure. We briefly recall that a PN structure on a differentiable manifold  $M$  is a pair  $(\pi_0, N)$ , where  $\pi_0$  is a Poisson bivector on  $M$  and  $N$  is an endomorphism of the tangent bundle  $TM$  whose Nijenhuis torsion vanishes and which satisfies a suitable compatibility condition with  $\pi_0$  [26]. With these ingredients one may introduce a second Poisson bivector,  $\pi_1 = \pi_0 \circ N$ , such that  $[\pi_0, \pi_1] = 0$ , where  $[\cdot, \cdot]$  is the Schouten bracket on polyvector fields. In a number of important cases — e.g. for the Calogero-Moser system [3] — the manifold  $M$  is a cotangent bundle,  $M = T^*X$ ,  $\pi_0$  is the inverse of the canonical symplectic form on  $M$ , and the recursion operator  $N = \pi_1 \circ \pi_0^{-1}$  turns out to be the *complete lift* of a torsionless endomorphism  $L: TX \rightarrow TX$  [34].

The notion of Poisson bracket has been recently generalized to a noncommutative geometric setting along the lines of the general approach introduced by Kontsevich [20] and developed by Ginzburg [15] and other authors in the symplectic case. In particular, a notion of *double Poisson structure* on a general associative noncommutative algebra  $A$  has been introduced by van den Bergh [35]. When  $A$  is the path algebra of a quiver an alternative, yet equivalent, definition has been proposed by Bielawski in the paper [4], where many explicit examples are discussed. Double Poisson structures on free associative algebras have been studied by Odesskii, Rubtsov and Sokolov [28], focusing in particular on linear and quadratic structures.

In this paper we make a further step in this direction by introducing and studying noncommutative Poisson-Nijenhuis structures on the path algebra  $A$  of a quiver  $Q$ . As well known, the algebra of noncommutative differential forms on  $A$  can be defined according to a universal construction valid for any associative algebra [19, 23]. On the other hand, a convenient notion of polyvector fields on  $A$  has been introduced in [4]: in this formalism a double Poisson structure on  $Q$  is equivalent to the assignment of a bivector  $\pi$  such that  $[\pi, \pi] = 0$  (see for details §§ 2.1, 2.2). The delicate issue is then to devise an appropriate definition of tensors of type  $(1, 1)$ , in order to have “recursion operators” as in the commutative setting (def. 8).

Once a Poisson bivector  $\pi$  and a recursion operator  $N$  on the path algebra  $A$  are given, one may mimic the classical theory of PN manifolds by noticing that all relevant results can be proved

in the purely algebraic language of Lie algebras and their deformations [21]. Along these lines, we are able to obtain theorem 14 generalizing the result concerning the existence of a hierarchy of compatible Poisson structures on any PN manifold. When the Poisson bivector arises from a noncommutative symplectic structure in the sense of [15], we prove theorem 16, which extends the usual result for  $\omega N$  manifolds. Furthermore, we are able to recover, in our environment, the above mentioned construction of  $\omega N$  manifolds through the complete lift of an endomorphism of the tangent bundle (§ 2.4).

In section 3 we discuss two significant applications of our formalism, namely the noncommutative versions of the rational Calogero-Moser system and of the Gibbons-Hernsen system. The bihamiltonian structure of the Calogero-Moser system was first described in [25]; more recently, a geometric interpretation of that structure was given in [3] by means of a two-step reduction of the two Poisson bivectors. The path algebra of the quiver with two loops provides the natural noncommutative counterpart of Calogero-Moser phase space, as shown in [15]. In § 3.1 we define a noncommutative  $\omega N$  structure on this path algebra and prove that it induces — first on the representation spaces, then on the quotient space — the  $\omega N$  structures used in [3].

The Gibbons-Hernsen system [14] is a generalization of Calogero-Moser but, up to our knowledge, no bihamiltonian structure for it is known. As a noncommutative counterpart of the rank 2 Gibbons-Hernsen phase space we take the path algebra of the double of the quiver (40) already studied by Bielawski and Pidstrygach in [5]. In § 3.2 we construct a noncommutative  $\omega N$  structure on this path algebra and obtain the corresponding bihamiltonian hierarchy. We expect that a corresponding  $\omega N$  structure is induced not only on the representation spaces of the quiver (40), but also on the phase space of the system (conjecturally to be defined along the same guidelines as in [3]). Finally, in section 4 we speculate briefly about other possible developments of the ideas presented in this paper.

In the remainder of this introduction we set up our notation for quivers and quiver representations (for this matter our basic reference is [7]).

## 1.1 Quivers and their representations

A quiver  $Q$  is a finite oriented graph. We think of  $Q$  as the (finite) set of its arrows; the (finite) set of vertices of  $Q$  will be denoted by  $I$  and its element will be labeled by  $e_1, \dots, e_n$ . One has maps  $h, t: Q \rightarrow I$  which associate to each arrow its head and tail. The double of  $Q$  is the quiver  $\overline{Q}$  obtained by attaching, for each arrow  $a$  in  $Q$ , a dual arrow  $a^*$  with the same endpoints but with opposite direction, that is  $t(a^*) = h(a)$ ,  $h(a^*) = t(a)$ .

Let  $\mathbb{k}$  be a field of characteristic zero. The path algebra  $\mathbb{k}Q$  is the associative algebra over  $\mathbb{k}$  generated by the paths in  $Q$  (including the trivial ones) with product given by concatenation of paths whenever is possible, zero otherwise. Clearly, the arrows  $\{a\}_{a \in Q}$  and the trivial paths, identified with the vertices  $e_1, \dots, e_n$ , are a set of generators for  $\mathbb{k}Q$ . If  $h(a) = t(b)$ , we shall write  $ba$  for the resulting concatenated path; observe that  $e_{h(a)}a = ae_{t(a)} = a$  for all  $a \in Q$ .

Let  $B$  denote the commutative semisimple algebra  $\bigoplus_{i=1}^n \mathbb{k}e_i$ , where the  $e_i$  are orthogonal idempotents, i.e.  $e_i^2 = 0$  and  $e_i e_j = 0$  for  $i \neq j$ . There is a natural algebra embedding  $B \hookrightarrow \mathbb{k}Q$  which gives  $\mathbb{k}Q$  a structure of  $B$ -algebra.

A  $\mathbb{k}$ -representation of a quiver  $Q$  is a pair  $(V, \tau)$ , where  $V = \bigoplus_{i \in I} V_i$  is an  $I$ -graded  $\mathbb{k}$ -vector space and  $\tau = (\tau_a)_{a \in Q}$  is a set of linear maps  $\tau_a \in \text{Hom}_{\mathbb{k}}(V_{t(a)}, V_{h(a)})$ . The space of the representations of  $Q$  on  $V$  will be denoted by  $\text{Rep}_{\mathbb{k}}(Q, V)$ .

Let us write  $\pi_i: V \rightarrow V_i$  for the canonical projection onto  $V_i$  and  $j_i: V_i \rightarrow V$  for the canonical immersion of  $V_i$ . Then each  $\tau_a$  determines an element  $\tilde{\tau}_a \in \text{End}(V)$  given by  $\tilde{\tau}_a = j_a \tau_a \pi_a$ ; similarly, for each vertex  $e_i$  we define  $\tilde{\tau}_i \in \text{End}(V)$  as the composition  $\tilde{\tau}_i = j_i \pi_i$ . It is straightforward

to verify that these endomorphisms satisfy the relations

$$\tilde{\tau}_i^2 = \tilde{\tau}_i; \quad \tilde{\tau}_i \tilde{\tau}_j = 0 \text{ for } i \neq j; \quad \tilde{\tau}_{h(a)} \tilde{\tau}_a = \tilde{\tau}_a \tilde{\tau}_{t(a)} = \tilde{\tau}_a.$$

The algebra  $\bigoplus_{i=1}^n \mathbb{k} \tilde{\tau}_i$  may be identified with  $B$ . Each representation  $(V, \tau)$  induces a  $B$ -algebra homomorphism  $\mathbb{k}Q \rightarrow \text{End}(V)$  defined by  $a \mapsto \tilde{\tau}_a$ ,  $e_i \mapsto \tilde{\tau}_i$ , and, conversely, each such a homomorphism determines a representation of  $Q$  on  $V$ . Summing up, one has an isomorphism

$$\mathcal{R}: \text{Rep}_{\mathbb{k}}(Q, V) \xrightarrow{\sim} \text{Hom}_{B\text{-alg}}(\mathbb{k}Q, \text{End}(V)). \quad (1)$$

Let us fix an element  $\mathbf{n} = (n_i)_{i \in I} \in \mathbb{N}^I$  and set  $|\mathbf{n}| = \sum_i n_i$ . The space of representations of the quiver  $Q$  on  $\mathbb{k}^{|\mathbf{n}|} = \bigoplus_{i \in I} \mathbb{k}^{n_i}$  will be denoted by  $\text{Rep}_{\mathbb{k}}(Q, \mathbf{n})$ :

$$\text{Rep}_{\mathbb{k}}(Q, \mathbf{n}) = \bigoplus_{a \in Q} \text{Mat}_{n_{h(a)} \times n_{t(a)}}(\mathbb{k}). \quad (2)$$

In this case the map (1) becomes

$$\begin{aligned} \mathcal{R}: \text{Rep}_{\mathbb{k}}(Q, \mathbf{n}) &\xrightarrow{\sim} \text{Hom}_{B\text{-alg}}(\mathbb{k}Q, \text{Mat}_{|\mathbf{n}| \times |\mathbf{n}|}(\mathbb{k})) \\ \tau &\longmapsto \mathcal{R}(\tau), \end{aligned} \quad (3)$$

where  $\mathcal{R}(\tau)(a) = \tilde{\tau}_a$  for all  $a \in Q$  and  $\mathcal{R}(\tau)(e_i) = \tilde{\tau}_i$  for all trivial paths  $e_i$ . Clearly, if we give a matrix  $R \in \text{Mat}_{|\mathbf{n}| \times |\mathbf{n}|}(\mathbb{k})$  the block decomposition  $R = R_{ij}$ , with  $R_{ij} \in \text{Mat}_{n_i \times n_j}(\mathbb{k})$ , the only non-zero block of  $\tilde{\tau}_a$  is the  $n_{h(a)} \times n_{t(a)}$  block corresponding to  $\tau_a$ , and the only non-zero block of  $\tilde{\tau}_i$  is the  $n_i \times n_i$  identity matrix.

The group

$$\text{GL}_{\mathbf{n}}(\mathbb{k}) := \prod_{i \in I} \text{GL}_{n_i}(\mathbb{k}) \quad (4)$$

acts naturally on  $\text{Rep}_{\mathbb{k}}(Q, \mathbf{n})$  by conjugation and preserves the decomposition (2). The subgroup

$$\mathbb{k}^* I_{\mathbf{n}} = \{(\lambda I_{n_i})_{i \in I} \mid \lambda \in \mathbb{k}^*\}$$

is contained in the center of  $\text{GL}_{\mathbf{n}}(\mathbb{k})$  and acts trivially on  $\text{Rep}_{\mathbb{k}}(Q, \mathbf{n})$ . Thus the action on  $\text{GL}_{\mathbf{n}}(\mathbb{k})$  factors through an action of the group

$$G_{\mathbf{n}} := \text{GL}_{\mathbf{n}}(\mathbb{k}) / \mathbb{k}^* I_{\mathbf{n}}. \quad (5)$$

The isomorphism classes of representations of the quiver  $Q$  with a fixed dimension vector  $\mathbf{n} = (\dim V_i)_{i \in I}$  are then in one to one correspondence with the set of orbits of  $G_{\mathbf{n}}$  in  $\text{Rep}_{\mathbb{k}}(Q, \mathbf{n})$ .

## 2 Non-commutative PN structures

### 2.1 General setting

In order to develop a noncommutative PN formalism on quiver path algebras we need to briefly recall some basic notions (see also [33] for a more pedagogical introduction).

Let  $A$  be a noncommutative, associative, unital algebra over a field  $\mathbb{k}$  of characteristic zero. The definition of the differential graded (DG) algebra of differential forms on  $A$  dates back to the classical work of A. Connes and M. Karoubi in the mid 1980s [19, 23, 8]. Let  $\tilde{A}$  be the quotient vector space  $A/\mathbb{k}$  and define

$$\Omega_{\mathbb{k}}^r(A) = A \otimes_{\mathbb{k}} \underbrace{\tilde{A} \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} \tilde{A}}_{r \text{ times}}$$

for any integer  $r \geq 0$ . The graded vector space  $\Omega_{\mathbb{k}}^{\bullet}(A) = \bigoplus_{r \geq 0} \Omega_{\mathbb{k}}^r(A)$  is endowed with the graded product

$$[a_0 \otimes a_1 \otimes \cdots \otimes a_r][a_{r+1} \otimes \cdots \otimes a_s] = \sum_{i=0}^r (-1)^{r-i} [a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_s], \quad (6)$$

where  $[a_0 \otimes a_1 \otimes \cdots \otimes a_r]$  is the class of  $a_0 \otimes a_1 \otimes \cdots \otimes a_r$  in  $\Omega_{\mathbb{k}}^r(A)$ , and with the differential

$$d[a_0 \otimes a_1 \otimes \cdots \otimes a_r] = [1 \otimes a_0 \otimes a_1 \otimes \cdots \otimes a_r]. \quad (7)$$

It is not difficult to show that these formulas determine the unique DG algebra structure on  $\Omega_{\mathbb{k}}^{\bullet}(A)$  satisfying the condition

$$[a_0 \otimes a_1 \otimes \cdots \otimes a_r] = a_0 da_1 \cdots da_r.$$

The mapping  $a_0 da_1 \mapsto a_0 \otimes a_1 - a_0 a_1 \otimes 1$  yields a natural isomorphism  $\Omega_{\mathbb{k}}^1(A) \xrightarrow{\sim} \ker \mu$ , where  $\mu: A \otimes_{\mathbb{k}} A \rightarrow A$  is the multiplication morphism. In this way  $\Omega_{\mathbb{k}}^1(A)$  can be given a structure of  $A$ -bimodule; while the left multiplication is the obvious one, the right multiplication is somewhat less evident:  $(a_0 da_1)a = a_0 d(a_1 a) - a_0 a_1 da$ .

The derivation functor  $\text{Der}_{\mathbb{k}}(A, \cdot): \mathbf{A}\text{-}\mathbf{Bimod} \rightarrow \mathbf{Vect}_{\mathbb{k}}$  is representable by  $\Omega_{\mathbb{k}}^1(A)$ . So, for any  $A$ -bimodule  $M$ , there is an isomorphism

$$\text{Der}_{\mathbb{k}}(A, M) \xrightarrow{\sim} \text{Hom}_{A\text{-Bimod}}(\Omega_{\mathbb{k}}^1(A), M).$$

When  $M = A$  this isomorphism induces a pairing

$$\begin{aligned} \Omega_{\mathbb{k}}^1(A) \times \text{Der}_{\mathbb{k}}(A, A) &\rightarrow A \\ (\alpha, \theta) &\mapsto i_{\theta}(\alpha) \end{aligned} \quad (8)$$

Notice that, since the linear space  $\text{Der}_{\mathbb{k}}(A, A)$  has no natural structure of  $A$ -bimodule<sup>1</sup>, this is just a pairing between vector spaces over  $\mathbb{k}$ . For any derivation  $\theta \in \text{Der}_{\mathbb{k}}(A, A)$  the operation  $i_{\theta}$  extends to the whole of  $\Omega_{\mathbb{k}}^{\bullet}(A)$ :

$$i_{\theta}(a_0 da_1 \cdots da_r) = \sum_{j=1}^r (-1)^{j-1} a_0 da_1 \cdots i_{\theta}(a_j) \cdots da_r.$$

The Lie derivative  $\mathcal{L}_{\theta}: \Omega_{\mathbb{k}}^{\bullet}(A) \rightarrow \Omega_{\mathbb{k}}^{\bullet}(A)$  with respect to  $\theta$  may then be defined using the Cartan formula  $\mathcal{L}_{\theta} = d \circ i_{\theta} + i_{\theta} \circ d$ . It follows that any Lie derivative  $\mathcal{L}_{\theta}$  is a degree zero derivation of  $\Omega_{\mathbb{k}}^{\bullet}(A)$ , and the following identities are readily verified on  $\Omega_{\mathbb{k}}^1(A)$  (and therefore on the whole of  $\Omega_{\mathbb{k}}^{\bullet}(A)$ ):

$$[\mathcal{L}_{\theta}, \mathcal{L}_{\eta}] = \mathcal{L}_{[\theta, \eta]}, \quad [\mathcal{L}_{\theta}, i_{\eta}] = i_{[\theta, \eta]}, \quad (9)$$

where  $[X, Y] = X \circ Y - Y \circ X$  is the usual commutator of endomorphisms.

The DG algebra  $\Omega_{\mathbb{k}}^{\bullet}(A)$  comes naturally equipped with the graded commutator

$$[\![\chi, \omega]\!] = \chi \omega - (-1)^{|\chi||\omega|} \omega \chi.$$

The abelianization of  $\Omega_{\mathbb{k}}^{\bullet}(A)$  is the graded vector space

$$\text{DR}_{\mathbb{k}}^{\bullet}(A) := \Omega_{\mathbb{k}}^{\bullet}(A) / [\![\Omega_{\mathbb{k}}^{\bullet}(A), \Omega_{\mathbb{k}}^{\bullet}(A)]\!],$$

---

<sup>1</sup>In general,  $\text{Der}_{\mathbb{k}}(A, A)$  is only a  $Z(A)$ -bimodule,  $Z(A)$  being the center of  $A$ . For quiver path algebras one has  $Z(A) = \mathbb{k}$ .

where  $[\![\Omega_{\mathbb{k}}^{\bullet}(A), \Omega_{\mathbb{k}}^{\bullet}(A)]\!]$  is the linear subspace generated by all graded commutators. The differential (7) descends to this quotient and so one gets a complex  $(\mathrm{DR}_{\mathbb{k}}^{\bullet}(A), d)$ , whose cohomology is, by definition, the noncommutative de Rham cohomology of  $A$ . Notice that, being every element of  $A$  of degree zero, one has  $[\![A, A]\!] = [A, A]$  so that the degree zero term of this complex is the linear space  $\mathrm{DR}_{\mathbb{k}}^0(A) = A/[A, A]$ , to be interpreted as the space of “regular functions” associated to the algebra  $A$ . Similarly, the degree one term is  $\mathrm{DR}_{\mathbb{k}}^1(A) = \Omega_{\mathbb{k}}^1(A)/[A, \Omega_{\mathbb{k}}^1(A)]$ .

It is easy to verify that, for any derivation  $\theta \in \mathrm{Der}_{\mathbb{k}}(A, A)$ , the operations  $i_{\theta}$  and  $\mathcal{L}_{\theta}$  induce operations, denoted by the same symbols, on the complex  $\mathrm{DR}_{\mathbb{k}}^{\bullet}(A)$ . We can therefore define a linear pairing  $\langle \cdot, \cdot \rangle: \mathrm{DR}_{\mathbb{k}}^1(A) \times \mathrm{Der}_{\mathbb{k}}(A, A) \rightarrow \mathrm{DR}_{\mathbb{k}}^0(A)$  given by

$$\langle \alpha, \theta \rangle = i_{\theta}(\alpha) \mod [A, A]. \quad (10)$$

Whenever a subalgebra  $B \hookrightarrow A$  is assigned, all previous constructions can be performed relatively to  $B$ . Specifically, one sets

$$\Omega_B^r(A) = A \otimes_B \underbrace{A/B \otimes_B \cdots \otimes_B A/B}_{r \text{ times}}, \quad \Omega_B^{\bullet}(A) = \bigoplus_{r \geq 0} \Omega_B^r(A)$$

and checks that the formulas (6), (7) descend to  $\Omega_B^{\bullet}(A)$  and endow it with a structure a DG algebra. The vector space  $\Omega_B^1(A)$  is isomorphic to the kernel of the multiplication morphism  $A \otimes_B A \rightarrow A$  (thus inheriting a structure of  $A$ -bimodule) and represents the derivation functor  $\mathrm{Der}_B(A, \cdot): A\text{-}\mathbf{Bimod} \rightarrow \mathbf{Vect}_{\mathbb{k}}$ . The relative de Rham complex of  $A$  is then defined as the quotient

$$\mathrm{DR}_B^{\bullet}(A) = \Omega_B^{\bullet}(A)/[\![\Omega_B^{\bullet}(A), \Omega_B^{\bullet}(A)]\!]$$

and, as expected, one has a pairing

$$\langle \cdot, \cdot \rangle: \mathrm{DR}_B^1(A) \times \mathrm{Der}_B(A, A) \rightarrow \mathrm{DR}_B^0(A). \quad (11)$$

## 2.2 Differential calculus on path algebras

From now on we shall restrict our attention to the case when  $A$  is the path algebra  $\mathbb{k}Q$  of a quiver  $Q$  and  $B = \bigoplus_{i \in I} \mathbb{k}e_i$  is its commutative subalgebra of idempotents. To make the notation less cumbersome, we shall adopt the following abbreviations:

$$\Omega^{\bullet}(Q) := \Omega_B^{\bullet}(\mathbb{k}Q), \quad \mathrm{Der}(Q) := \mathrm{Der}_B(\mathbb{k}Q, \mathbb{k}Q), \quad \mathrm{DR}^{\bullet}(Q) := \mathrm{DR}_B^{\bullet}(\mathbb{k}Q).$$

Following R. Bielawski’s approach [4], we denote each dual arrow  $a^*$  of the double quiver  $\overline{Q}$  by  $\partial_a$  and think of it as a fundamental noncommutative vector field. To emphasize this different interpretation of  $\overline{Q}$  we adopt a new symbol to denote it:  $\mathrm{T}Q$ .

Let us consider the linear subspace  $\mathbb{k}\mathrm{T}Q^r \subset \mathbb{k}\mathrm{T}Q$  generated by all the monomials  $x_1 \cdots x_k$  with  $k \geq r$  such that exactly  $r$  of the  $x_i$  are of the type  $\partial_a$  for some  $a \in Q$ . Obviously, one has  $\mathbb{k}\mathrm{T}Q^0 = \mathbb{k}Q$ . The vector space  $\mathbb{k}\mathrm{T}Q$  can therefore be given the grading

$$\mathbb{k}\mathrm{T}Q = \bigoplus_{r \geq 0} \mathbb{k}\mathrm{T}Q^r. \quad (12)$$

**Definition 1.** The space  $\mathcal{V}Q$  of *noncommutative polyvector fields* on the quiver  $Q$  is the quotient of  $\mathbb{k}\mathrm{T}Q$  by the relations

$$PR - (-1)^{pr} RP = 0, \quad \text{if } P \in \mathbb{k}\mathrm{T}Q^p, R \in \mathbb{k}\mathrm{T}Q^r. \quad (13)$$

It is worth observing that every path which is not closed becomes zero in  $\mathcal{V}Q$ ; in other words,  $\mathcal{V}Q$  is generated by closed paths (“necklaces”). The grading (12) induces a grading on  $\mathcal{V}Q$ , i.e.  $\mathcal{V}Q = \bigoplus_{r \geq 0} \mathcal{V}^r Q$ . Notice that  $\mathcal{V}^0 Q = \text{DR}^0(Q)$ . As for  $\mathcal{V}^1 Q$ , its elements can be written in the canonical form

$$\theta = \sum_{a \in Q} p_a \partial_a, \quad \text{with } p_a \in \mathbb{k}Q, \quad e_{h(a)} p_a = p_a, \quad p_a e_{t(a)} = p_a. \quad (14)$$

**Lemma 2.** *There is a canonical isomorphism  $\mathcal{V}^1 Q \simeq \text{Der}(Q)$ .*

*Proof.* Each element  $\theta \in \mathcal{V}^1 Q$  of the form (14) uniquely determines a  $B$ -linear derivation  $A \rightarrow A$  defined by mapping each arrow  $a$  to the path  $p_a$  and each idempotent  $e_i$  to zero.  $\square$

A canonical form is also available (see e.g. [6]) for every 1-form  $\alpha \in \text{DR}^1(Q)$ :

$$\alpha = \sum_{a \in Q} r_a da, \quad \text{with } r_a \in \mathbb{k}Q, \quad e_{t(a)} r_a = r_a, \quad r_a e_{h(a)} = r_a. \quad (15)$$

Using expressions (14) and (15) the pairing  $\langle \cdot, \cdot \rangle: \text{DR}^1(Q) \times \text{Der}(Q) \rightarrow \text{DR}^0(Q)$  introduced in equation (11) becomes simply

$$\langle \alpha, \theta \rangle = \sum_{a \in Q} r_a p_a. \quad (16)$$

This pairing is “perfect” in the sense that  $\langle da, \partial_b \rangle = \delta_{ab}$  (but notice that both  $\text{Der}(Q)$  and  $\text{DR}^1(Q)$  are actually infinite-dimensional linear spaces over  $\mathbb{k}$ ).

The space  $\mathcal{V}Q$  of noncommutative polyvector fields can be endowed with a Schouten bracket [4, 29, 22]. For any arrow  $y \in \text{TT}Q$  and for any monomial  $x_1 \cdots x_N$ , with  $x_i \in \mathbb{k}\text{TT}Q$ , let

$$D_y(x_1 \cdots x_N) = \sum_{x_i=y} (-1)^{n_i m_i} x_{i+1} \cdots x_N x_1 \cdots x_{i-1}, \quad (17)$$

where  $n_i$  (resp.  $m_i$ ) is the number of dual arrows  $\partial_a$  among the elements  $x_1, \dots, x_i$  (resp. among  $x_{i+1}, \dots, x_N$ ). This operation can be extended linearly to the whole of  $\mathbb{k}\text{TT}Q$ , so obtaining a directional superderivative

$$D_y: \mathcal{V}Q \rightarrow \mathbb{k}\text{TT}Q.$$

**Definition 3.** Given  $\lambda \in \mathcal{V}^p Q$ ,  $\xi \in \mathcal{V}^q Q$ , their *Schouten bracket*  $[\lambda, \xi]$  is defined by the formula

$$[\lambda, \xi] = \sum_{a \in Q} D_{\partial_a}(\lambda) D_a(\xi) - (-1)^{(p+1)(q+1)} D_{\partial_a}(\xi) D_a(\lambda) \quad \text{modulo relations (13).}$$

For any  $\lambda \in \mathcal{V}^p Q$ ,  $\xi \in \mathcal{V}^q Q$ ,  $\sigma \in \mathcal{V}^r Q$ , the following properties hold true:

- 1)  $[\lambda, \xi] \in \mathcal{V}^{p+q-1} Q$ ;
- 2)  $[\lambda, \xi] = -(-1)^{(p+1)(q+1)} [\xi, \lambda]$ ;
- 3) (graded Jacobi identity)

$$[\lambda, [\xi, \sigma]] + (-1)^{(p+1)(q+1)} [\xi, [\sigma, \lambda]] + (-1)^{(q+1)(r+1)} [\sigma, [\lambda, \xi]] = 0.$$

Let us now consider, for a given dimension vector  $\mathbf{n}$ , the representation space  $\text{Rep}_{\mathbb{k}}(Q, \mathbf{n})$  of the quiver  $Q$  and denote its space of  $G_{\mathbf{n}}$ -invariant differential forms by  $\Omega^{\bullet}(\text{Rep}_{\mathbb{k}}(Q, \mathbf{n}))^{G_{\mathbf{n}}}$  and that of  $G_{\mathbf{n}}$ -invariant ordinary polyvector fields by  $\mathcal{V}(\text{Rep}_{\mathbb{k}}(Q, \mathbf{n}))^{G_{\mathbf{n}}}$  (the group  $G_{\mathbf{n}}$  is defined in eq. (5)). The space  $\mathcal{V}(\text{Rep}_{\mathbb{k}}(Q, \mathbf{n}))^{G_{\mathbf{n}}}$  comes equipped with the bracket induced by the usual Schouten bracket on the space  $\mathcal{V}(\text{Rep}_{\mathbb{k}}(Q, \mathbf{n}))$ , namely

$$\begin{aligned} [X_1 \wedge \cdots \wedge X_p, Y_1 \wedge \cdots \wedge Y_q] = \\ = \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge X_{i-1} \wedge X_{i+1} \wedge \cdots \wedge X_p \wedge Y_1 \wedge \cdots \wedge Y_{j-1} \wedge Y_{j+1} \wedge \cdots \wedge Y_q. \end{aligned}$$

**Theorem 4.** *Let  $\text{Rep}_{\mathbb{k}}(Q, \mathbf{n})$  be a representation space for the quiver  $Q$ .*

1) *There is a morphism of graded  $B$ -algebras*

$$\wedge : \text{DR}^{\bullet}(Q) \rightarrow \Omega^{\bullet}(\text{Rep}_{\mathbb{k}}(Q, \mathbf{n}))^{G_{\mathbf{n}}}$$

*which commutes with the respective differentials;*

2) *there is a morphism of graded  $B$ -algebras*

$$\smile : \mathcal{V}Q \rightarrow \mathcal{V}(\text{Rep}_{\mathbb{k}}(Q, \mathbf{n}))^{G_{\mathbf{n}}}$$

*which commutes with the respective Schouten brackets;*

3) *for every  $\alpha \in \text{DR}^1(Q)$  and  $\theta \in \mathcal{V}^1Q$  one has*

$$\widehat{\langle \alpha, \theta \rangle} = \langle \hat{\alpha}, \check{\theta} \rangle.$$

*Proof.* A proof of (1) can be found in V. Ginzburg's *Lectures* [16, §12.6] in the case of a general noncommutative associative algebra. Point (2) is proved in [4]; point (3) is then straightforward in view of lemma 2.  $\square$

The formalism we have set up makes it natural to state the following definition [4].

**Definition 5.** A *double Poisson structure* on  $Q$  is a noncommutative bivector  $\pi \in \mathcal{V}^2Q$  such that  $[\pi, \pi] = 0$ .

As an immediate consequence of theorem 4 we get the following result, which is crucial for the applications we shall describe in section 3.

**Corollary 6.** *If  $\pi$  is a double Poisson structure on  $Q$  then  $\tilde{\pi}$  is a Poisson structure on each representation space  $\text{Rep}_{\mathbb{k}}(Q, \mathbf{n})$ .*

The previous corollary has a converse, which shows that a double Poisson structure on  $Q$  is completely determined by the family of all induced Poisson structures on the representation spaces of  $Q$ .

**Theorem 7** (Theorem 3.9 in [4]). *Let  $\pi \in \mathcal{V}^2Q$ . If  $\tilde{\pi}$  is a Poisson structure on all representation spaces  $\text{Rep}_{\mathbb{k}}(Q, \mathbf{n})$  then  $\pi$  is a double Poisson structure on  $\mathbb{k}Q$ .*

In view of the sequel we need to clarify the relationship between noncommutative bivectors, that is elements of  $\mathcal{V}^2Q$ , and linear maps  $\text{DR}^1(Q) \rightarrow \text{Der}(Q)$ .

Without loss of generality we can write a bivector  $\pi \in \mathcal{V}^2 Q$  as a sum of ordinary (i.e. not graded) commutators of the form

$$\pi = \sum_{a,b \in Q} \sum_{j \in J} [P_j^{ab} \partial_a, R_j^{ab} \partial_b],$$

where  $J$  is some finite set and  $P_j^{ab}, R_j^{ab}$  are paths in the quiver  $Q$  such that each resulting monomial is a closed path in  $\text{T}Q$ . We define the corresponding map  $\tilde{\pi}: \text{DR}^1(Q) \rightarrow \text{Der}(Q)$  in the following way: given  $\alpha \in \text{DR}^1(Q)$ , let  $\sum_{c \in Q} S_c dc$  be a representative for  $\alpha$  in canonical form. Then

$$\tilde{\pi}(\alpha) := \sum_{a,b \in Q} \sum_{j \in J} (P_j^{ab} S_a R_j^{ab} \partial_b - R_j^{ab} S_b P_j^{ab} \partial_a).$$

This works because for each arrow  $a$  the path  $S_a$  runs in the opposite direction to  $a$ , hence it can always replace  $\partial_a$  inside a word in  $\text{kT}Q$  without making it zero.

We can relate the map  $\tilde{\pi}$  with the action of the bivector  $\pi \in \mathcal{V}^2 Q$  on a pair of 2-forms  $\alpha, \beta \in \text{DR}^1(Q)$  by the usual formula

$$\pi(\alpha, \beta) = \langle \beta, \tilde{\pi}(\alpha) \rangle = i_{\tilde{\pi}(\alpha)}(\beta). \quad (18)$$

More explicitly, if  $\alpha$  is represented by  $\sum_c S_c dc$  and  $\beta$  by  $\sum_{c'} T_{c'} dc'$  then

$$\pi(\alpha, \beta) = \sum_{a,b \in Q} \sum_{j \in J} (P_j^{ab} S_a R_j^{ab} T_b - R_j^{ab} S_b P_j^{ab} T_a).$$

Now we would like to introduce the analogue of tensors of type  $(1, 1)$  on  $\text{k}Q$ . We locate the salient feature of these objects in their ability to be interpreted simultaneously as maps  $N: \text{Der}(Q) \rightarrow \text{Der}(Q)$  and as maps  $N^*: \text{DR}^1(Q) \rightarrow \text{DR}^1(Q)$ , related by the familiar equality

$$\langle N^*(\alpha), \theta \rangle = \langle \alpha, N(\theta) \rangle \quad \text{for every } \alpha \in \text{DR}^1(Q), \theta \in \text{Der}(Q), \quad (19)$$

where the pairing is defined by equation (16). Not every endomorphism of  $\text{Der}(Q)$  has this property.

**Definition 8.** A  $\text{k}$ -linear endomorphism  $N: \text{Der}(Q) \rightarrow \text{Der}(Q)$  is called *regular* if there exists a derivation  $d^N: \text{k}Q \rightarrow \Omega^1(Q)$  such that  $i_\theta \circ d^N = N(\theta)$  for every  $\theta \in \text{Der}(Q)$ .

Clearly the map  $d^N$ , if exists, is unique since for every arrow  $a \in Q$  the 1-form  $d^N a$  is completely determined by the paths  $(i_{\partial_b}(d^N a))_{b \in Q}$ , which by definition coincide with  $N(\partial_b)(a)$ . It follows that to each regular endomorphism  $N$  we can associate the unique morphism of  $\text{k}Q$ -bimodules

$$N^*: \Omega^1(Q) \rightarrow \Omega^1(Q)$$

defined by sending each generator  $da$  of  $\Omega^1(Q)$  to the 1-form  $d^N a$ . As every morphism of  $\text{k}Q$ -bimodules preserves the linear subspace  $[\text{k}Q, \Omega^1 Q]$  inside  $\Omega^1(Q)$ , this recipe induces a unique  $\text{k}$ -linear map  $\text{DR}^1(Q) \rightarrow \text{DR}^1(Q)$  that we also denote by  $N^*$ . By definition, we have

$$\langle N^*(da), \theta \rangle = i_\theta(N^*(da)) = i_\theta(d^N a) = N(\theta)(a) = i_{N(\theta)}(da) = \langle da, N(\theta) \rangle,$$

from which (19) follows by the  $\text{k}Q$ -linearity of  $N^*$  and  $i_\theta$ . We shall call  $N^*$  the *transpose* of  $N$ .

Let us remark that, conversely, if we are given a  $\text{k}Q$ -linear map  $N^*: \Omega^1(Q) \rightarrow \Omega^1(Q)$  such that (19) holds then the endomorphism  $N$  is necessarily regular, as the map  $d^N: \text{k}Q \rightarrow \Omega^1(Q)$  defined as the unique derivation sending the arrow  $a \in Q$  to  $N^*(da)$  clearly has the property required by definition 8.

Hence for us a *tensor of type (1,1) on  $\text{k}Q$*  will be given by, equivalently, a regular endomorphism  $N: \text{Der}(Q) \rightarrow \text{Der}(Q)$  or the corresponding transpose  $N^*: \text{DR}^1(Q) \rightarrow \text{DR}^1(Q)$ .



### 2.3 PN structures on path algebras

We continue by briefly recalling some concepts and results from [21].

**Definition 9.** Let  $E$  be a Lie algebra over the field  $\mathbb{k}$  and  $N: E \rightarrow E$  be a linear map.

1. The  $N$ -deformed bracket on  $E$  is the  $E$ -valued 2-form on  $E$  defined by

$$[x, y]_N := [N(x), y] + [x, N(y)] - N([x, y]).$$

2. The *Nijenhuis torsion* of  $N$  is the  $E$ -valued 2-form on  $E$  defined by

$$\mathcal{T}_N(x, y) := [N(x), N(y)] - N([N(x), y] + [x, N(y)] - N([x, y])). \quad (20)$$

Two Lie brackets on  $E$  are *compatible* if their sum is also a Lie bracket.

**Theorem 10** (Corollary 1.1 in [21]). *If  $\mathcal{T}_N = 0$  then  $[\cdot, \cdot]_N$  is also a Lie bracket on  $E$  which is compatible with  $[\cdot, \cdot]$ .*

It also follows that  $N$  is a morphism of Lie algebras from  $(E, [\cdot, \cdot]_N)$  to  $(E, [\cdot, \cdot])$ : indeed the condition  $\mathcal{T}_N = 0$  is equivalent to

$$[N(x), N(y)] = N([x, y]_N) \quad \text{for all } x, y \in E.$$

We can now give the following

**Definition 11.** Let  $Q$  be a quiver. A *Nijenhuis tensor on the path algebra*  $\mathbb{k}Q$  is a regular endomorphism  $N: \text{Der}(Q) \rightarrow \text{Der}(Q)$  such that  $\mathcal{T}_N = 0$ .

Let us recall how a Nijenhuis tensor on  $\mathbb{k}Q$  defines a “deformed version” of the Cartan calculus on  $\text{DR}^\bullet(Q)$ . We have already defined the derivation  $d^N: \mathbb{k}Q \rightarrow \Omega^1(Q)$ , which can be extended to a degree 1 derivation of the DG algebra  $\Omega^\bullet(Q)$  in the usual way, that is imposing the (anti)commutation rule

$$d^N \circ d + d \circ d^N = 0.$$

The vanishing of  $\mathcal{T}_N$  then implies  $d^N \circ d^N = 0$ . We can also define a deformed Lie derivative  $\mathcal{L}_\theta^N$  as

$$\mathcal{L}_\theta^N := d^N \circ i_\theta + i_\theta \circ d^N.$$

These operators obey a deformed version of the identities (9) where the usual commutator bracket on  $\text{Der}(Q)$  is replaced by the bracket  $[\cdot, \cdot]_N$ :

$$[\mathcal{L}_\theta^N, \mathcal{L}_\eta^N] = \mathcal{L}_{[\theta, \eta]_N}^N, \quad [\mathcal{L}_\theta^N, i_\eta] = i_{[\theta, \eta]_N}.$$

In particular the action of  $\mathcal{L}_\theta^N$  on 1-forms is given by

$$\mathcal{L}_\theta^N(\beta) = \mathcal{L}_{N(\theta)}(\beta) - \mathcal{L}_\theta(N^*(\beta)) + N^*(\mathcal{L}_\theta(\beta)).$$

Finally, both maps  $d^N$  and  $\mathcal{L}_\theta^N$  descend from  $\Omega^\bullet(Q)$  to  $\text{DR}^\bullet(Q)$  by the usual arguments.

Suppose now that the quiver  $Q$  comes equipped with a noncommutative bivector  $\pi \in \mathcal{V}^2 Q$ . Then we can “dualize” the Lie bracket  $[\cdot, \cdot]$  on  $\text{Der}(Q)$  to a bracket defined on  $\text{DR}^1(Q)$  according to the well-known formula

$$\{\alpha, \beta\}_\pi := \mathcal{L}_{\tilde{\pi}(\alpha)}(\beta) - \mathcal{L}_{\tilde{\pi}(\beta)}(\alpha) - d(\pi(\alpha, \beta)). \quad (21)$$

**Theorem 12** (Proposition 3.2 in [21]). *The bracket (21) obeys the Jacobi identity if and only if  $\pi$  is a double Poisson structure, and in this case one has*

$$\tilde{\pi}(\{\alpha, \beta\}_\pi) = [\tilde{\pi}(\alpha), \tilde{\pi}(\beta)], \quad (22)$$

that is,  $\tilde{\pi}$  is a morphism of Lie algebras.

*Proof.* The proof in [21] uses only the algebraic properties of the Schouten bracket, therefore it works also in our setting. Alternatively, one can simply observe that the analogous result holds for the induced (commutative) structures on every representation space, hence it must hold already at the noncommutative level.  $\square$

From now on we assume that the path algebra  $\mathbb{k}Q$  is equipped with both a Nijenhuis tensor  $N$  and a double Poisson structure  $\pi$ . Let us say that  $N$  and  $\pi$  are *algebraically compatible* if  $N \circ \tilde{\pi} = \tilde{\pi} \circ N^*$  as maps  $\text{DR}^1(Q) \rightarrow \text{Der}(Q)$ . We denote by  $\pi^N$  the (unique) bivector in  $\mathcal{V}^2 Q$  associated to this map; then for every pair of 1-forms  $\alpha, \beta \in \text{DR}^1(Q)$  we have

$$\pi^N(\alpha, \beta) = \pi(N^*(\alpha), \beta) = \pi(\alpha, N^*(\beta)). \quad (23)$$

Now let us introduce, again following [21], two possible deformations of the bracket (21). The first one is simply its  $N^*$ -deformed version, in the sense of definition 9:

$$\{\alpha, \beta\}_{\pi, N^*} = \{N^*(\alpha), \beta\}_\pi + \{\alpha, N^*(\beta)\}_\pi - N^*(\{\alpha, \beta\}_\pi). \quad (24)$$

The second deformation is obtained by replacing the operators  $d$  and  $\mathcal{L}_\theta$  in the definition (21) with the operators  $d^N$  and  $\mathcal{L}_\theta^N$  introduced above. The resulting bracket reads

$$\begin{aligned} \{\alpha, \beta\}'_\pi &= \mathcal{L}_{N(\tilde{\pi}(\alpha))}(\beta) - \mathcal{L}_{\tilde{\pi}(\alpha)}(N^*(\beta)) + N^*(\mathcal{L}_{\tilde{\pi}(\alpha)}(\beta)) + \\ &\quad - \mathcal{L}_{N(\tilde{\pi}(\beta))}(\alpha) + \mathcal{L}_{\tilde{\pi}(\beta)}(N^*(\alpha)) - N^*(\mathcal{L}_{\tilde{\pi}(\beta)}(\alpha)) - N^*(d(\pi(\alpha, \beta))). \end{aligned} \quad (25)$$

We can now define the noncommutative version of the *Magri-Morosi concomitant* as

$$C_{(\pi, N)}(\alpha, \beta) := \frac{1}{2} (\{\alpha, \beta\}_{\pi, N^*} - \{\alpha, \beta\}'_\pi).$$

By direct computation one sees that

$$C_{(\pi, N)}(\alpha, \beta) = \mathcal{L}_{\tilde{\pi}(\alpha)}(N^*(\beta)) - \mathcal{L}_{\tilde{\pi}(\beta)}(N^*(\alpha)) - d(\pi^N(\alpha, \beta)) - N^*(\mathcal{L}_{\tilde{\pi}(\alpha)}(\beta) - \mathcal{L}_{\tilde{\pi}(\beta)}(\alpha) - d(\pi(\alpha, \beta))).$$

We say that  $\pi$  and  $N$  are *differentially compatible* if  $C_{(\pi, N)} = 0$ ; obviously this happens if and only if the two brackets (24) and (25) coincide. As shown in [21] this also implies that both these brackets coincide with the bracket between 1-forms induced by the bivector  $\pi^N$  defined by (23).

**Definition 13.** Let  $Q$  be a quiver,  $\pi \in \mathcal{V}^2 Q$  a double Poisson structure and  $N: \text{Der}(Q) \rightarrow \text{Der}(Q)$  a Nijenhuis tensor on  $\mathbb{k}Q$ . We say that  $\pi$  and  $N$  are *compatible* if

1.  $N \circ \tilde{\pi} = \tilde{\pi} \circ N^*$  and
2.  $C_{(\pi, N)} = 0$ .

In this case we call the pair consisting of  $\pi$  and  $N$  a *Poisson-Nijenhuis structure on the path algebra  $\mathbb{k}Q$* .

In this new setting the main result of the theory reads as follows:

**Theorem 14.** *Let  $Q$  be a quiver and  $(\pi, N)$  a Poisson-Nijenhuis structure on  $\mathbb{k}Q$ . Then the bivector  $\pi^N$  defined by equation (23) is a double Poisson structure on  $\mathbb{k}Q$  which is compatible with  $\pi$ .*

*Proof.* As in the classical case [21] the result hinges on the following identity, which is valid for every double Poisson structure  $\pi$  and for every regular endomorphism  $N$  (not necessarily with zero torsion) which is algebraically compatible with it:

$$\langle \mathcal{T}_{N^*}(\alpha, \beta), \theta \rangle + \langle \alpha, \mathcal{T}_N(\tilde{P}(\beta), \theta) \rangle = \langle C_{(\pi, N)}(N^*(\alpha), \beta), \theta \rangle - \langle C_{(\pi, N)}(\alpha, \beta), N(\theta) \rangle. \quad (26)$$

This equality can be verified by a direct computation. In the hypotheses stated, identity (26) implies that  $\mathcal{T}_{N^*} = 0$ . By theorem 1 it follows that the bracket  $\{\cdot, \cdot\}_{\pi, N^*}$  obeys the Jacobi identity and is compatible with  $\{\cdot, \cdot\}_{\pi}$ . But the bracket  $\{\cdot, \cdot\}_{\pi, N^*}$  coincides with the bracket  $\{\cdot, \cdot\}_{\pi^N}$ , so that  $\pi^N$  is a double Poisson structure by theorem 2 and is compatible with  $\pi$ .  $\square$

Clearly the process can be iterated, so that on a quiver equipped with a PN structure we have a whole hierarchy of double Poisson structures defined by the maps  $(\tilde{\pi}_k)_{k \geq 0}$ , where

$$\tilde{\pi}_k := \underbrace{N \circ \cdots \circ N}_{k \text{ times}} \circ \tilde{\pi},$$

and every pair of such double Poisson structures is compatible.

An important special case is when the first Poisson structure is *invertible*, that is, when it comes from a noncommutative symplectic structure on  $\mathbb{k}Q$ . Let us recall [20, 15] that a *noncommutative symplectic structure* on  $\mathbb{k}Q$  is given by a 2-form  $\omega \in \text{DR}^2(Q)$  which is closed ( $d\omega = 0$ ) and non degenerate, meaning that the map  $\omega^\flat: \text{Der}(Q) \rightarrow \text{DR}^1(Q)$  defined by  $\theta \mapsto i_\theta(\omega)$  is invertible. Let us denote by  $\omega^\sharp: \text{DR}^1(Q) \rightarrow \text{Der}(Q)$  its inverse; it maps a 1-form  $\alpha$  to the unique derivation such that  $i_{\omega^\sharp(\alpha)}(\omega) = \alpha$ . For any  $f \in \text{DR}^0(Q)$  we have the corresponding “Hamiltonian derivation”  $\theta_f = -\omega^\sharp(df)$ .

The following lemma (already implicit in [4]) clarifies the relationship between symplectic forms and Poisson bivectors on quiver path algebras.

**Lemma 15.** *Suppose  $\omega$  is a non-degenerate 2-form on  $\mathbb{k}Q$  and let  $\pi \in \mathcal{V}^2Q$  be the bivector associated to  $-\omega^\sharp$  by the equality (18). Then  $\omega$  is symplectic ( $d\omega = 0$ ) if and only if  $\pi$  is a double Poisson structure ( $[\pi, \pi] = 0$ ).*

*Proof.* The 2-form  $\omega$  defines a bilinear, skew-symmetric bracket on  $\text{DR}^0(Q)$  by the usual prescription:

$$\{f, g\} := i_{\theta_g}(i_{\theta_f}(\omega)).$$

This induces a bilinear and skew-symmetric bracket on the space of  $G_{\mathbf{n}}$ -invariant functions defined on every representation space  $\text{Rep}_{\mathbf{k}}(Q, \mathbf{n})$ . By known results, the latter brackets obey the Jacobi identity if and only if the induced 2-forms  $\hat{\omega}$  are closed, and this happens if and only if  $d\omega = 0$ .

On the other hand, the bracket defined on  $\text{DR}^0(Q)$  by the noncommutative bivector  $\pi$  is the same as above, since

$$\pi(df, dg) = \langle dg, -\omega^\sharp(df) \rangle = i_{\theta_f}(dg) = i_{\theta_f}(-\omega^\flat(\theta_g)) = -i_{\theta_f}(i_{\theta_g}(\omega)) = i_{\theta_g}(i_{\theta_f}(\omega)).$$

It follows that the induced brackets on representation spaces obey the Jacobi identity if and only if the induced bivectors  $\tilde{\pi}$  are Poisson. By theorem 7 this happens if and only if  $[\pi, \pi] = 0$ .  $\square$

Hence every noncommutative symplectic structure on  $\mathbb{k}Q$  gives rise to a unique double Poisson structure on  $\mathbb{K}Q$ , exactly as in the classical setting. We also have the following analogue of another well-known result.

**Theorem 16.** *Suppose  $\omega$  is a noncommutative symplectic form on  $\mathbb{k}Q$  with associated double Poisson structure  $\pi_0$  and let  $\pi_1$  be another double Poisson structure on  $\mathbb{k}Q$ . Then:*

1. *the endomorphism  $N: \text{Der}(Q) \rightarrow \text{Der}(Q)$  defined by  $\tilde{\pi}_1 \circ (-\omega^\flat)$  is regular;*
2. *if  $\pi_0$  and  $\pi_1$  are compatible then  $N$  is Nijenhuis and compatible with  $\pi_0$ .*

*Proof.* (1) It suffices to show that there exists a map  $N^*: \text{DR}^1(Q) \rightarrow \text{DR}^1(Q)$  such that  $\langle N^*(\alpha), \theta \rangle$  equals

$$\langle \alpha, N(\theta) \rangle = -\langle \alpha, \tilde{\pi}_1(\omega^\flat(\theta)) \rangle = -\langle \alpha, \tilde{\pi}_1(i_\theta(\omega)) \rangle = -\pi_1(i_\theta(\omega), \alpha) = \pi_1(\alpha, i_\theta(\omega))$$

for any  $\alpha \in \text{DR}^1(Q)$ ,  $\theta \in \text{Der}(Q)$ . We claim that  $N^* := -\omega^\flat \circ \tilde{\pi}_1$  does the job: indeed,

$$\langle -\omega^\flat(\tilde{\pi}_1(\alpha)), \theta \rangle = -i_\theta(i_{\tilde{\pi}_1(\alpha)}(\omega)) = i_{\tilde{\pi}_1(\alpha)}(i_\theta(\omega)) = \langle i_\theta(\omega), \tilde{\pi}_1(\alpha) \rangle = \pi_1(\alpha, i_\theta(\omega)),$$

as we wanted.

(2) The first assertion follows from the following identity which, in the hypotheses stated, relates the torsion of  $N$  to the Schouten bracket of  $\pi_0$  and  $\pi_1$ :

$$\mathcal{T}_N(\theta, \eta) = 2N([\pi_0, \pi_1](\omega^\flat(\theta), \omega^\flat(\eta))).$$

By a direct computation one then shows that  $C_{(\pi, N)} = 0$ . □

As is customary, a PN structure in which one of the two Poisson bivectors comes from a symplectic form will be called a  $\omega N$  structure.

## 2.4 Noncommutative lifts

Now we would like to reinterpret in our setting the construction of compatible Poisson brackets on cotangent bundles introduced in [34] and used in [3] to obtain the bihamiltonian structure of the Calogero-Moser system.

The noncommutative analogue of cotangent bundles are double quivers, so let us consider a quiver  $Q$  and its double  $\overline{Q}$ , where for each arrow  $a$  we denote its opposite by  $a^*$ . We have the tautological 1-form  $\lambda \in \text{DR}^1(\overline{Q})$  represented by the expression  $\sum_{a \in Q} a^* da$  and the corresponding canonical symplectic form  $\omega = d\lambda$  in  $\text{DR}^2(\overline{Q})$ , represented by

$$\omega = \sum_{a \in Q} da^* da.$$

Consider now a regular endomorphism  $L: \text{Der}(Q) \rightarrow \text{Der}(Q)$  on the path algebra of  $Q$ . We define the following deformation of  $\lambda$ ,

$$\lambda_L := \sum_{a \in Q} a^* L^*(da) = \sum_{a \in Q} a^* d^L a,$$

and denote by  $\omega_L$  its differential (which is not a symplectic form in general).

The *complete lift* of  $L$  to the double  $\overline{Q}$  is the map  $N: \text{Der}(\overline{Q}) \rightarrow \text{Der}(\overline{Q})$  defined by

$$\theta \mapsto \omega^\sharp(i_\theta(\omega_L)). \tag{27}$$

In other words,  $N(\theta)$  is the unique derivation of  $\mathbb{k}\overline{Q}$  such that

$$i_{N(\theta)}(\omega) = i_\theta(\omega_L).$$

**Lemma 17.**  $N$  is a regular endomorphism of  $\text{Der}(\overline{Q})$ .

*Proof.* It suffices to show that  $N$  has a transpose. Let us write the generic 1-form in  $\text{DR}^1(\overline{Q})$  as  $\omega^b(\eta)$ , with  $\eta \in \text{Der}(\overline{Q})$ . We need a map  $N^*: \text{DR}^1(\overline{Q}) \rightarrow \text{DR}^1(\overline{Q})$  such that  $\langle N^*(\omega^b(\eta)), \theta \rangle$  equals

$$\langle \omega^b(\eta), N(\theta) \rangle = i_{N(\theta)}(i_\eta(\omega)) = -i_\eta(i_{N(\theta)}(\omega)) = -i_\eta(i_\theta(\omega_L)) = i_\theta(i_\eta(\omega_L))$$

for every  $\theta \in \text{Der}(\overline{Q})$ . Clearly then we should take

$$N^*(\omega^b(\eta)) = i_\eta(\omega_L),$$

that is,  $N^*(\beta) := i_{\omega^\sharp(\beta)}(\omega_L)$  for every  $\beta \in \text{DR}^1(\overline{Q})$ .  $\square$

Let us consider now the bivector defined by the map

$$\tilde{\pi}_1 := -N \circ \omega^\sharp.$$

Explicitly, one has

$$\pi_1(\alpha, \beta) = i_{\omega^\sharp(\beta)}(i_{\omega^\sharp(\alpha)}(\omega_L)),$$

as may be verified by a direct computation. This is the noncommutative version of the bivector considered in [34, 18].

**Theorem 18.** If  $\mathcal{T}_L = 0$  then the bivector  $\pi_1$  is Poisson and compatible with the canonical Poisson structure.

*Proof.* The noncommutative bivector  $\pi_1$  induces a genuine bivector  $\tilde{\pi}_1$  on every representation space  $\text{Rep}_{\mathbb{k}}(Q, \mathbf{n})$ , and consequently a bracket on  $G_{\mathbf{n}}$ -invariant regular functions. The proofs in [34] and [18] then show that these brackets obey the Jacobi identity when  $L$  is torsionless. This means that  $\pi_1$  induces a Poisson bivector on every representation space, hence theorem 7 implies that  $\pi_1$  is a double Poisson structure on  $\mathbb{k}\overline{Q}$ . Compatibility with  $\pi_0$  then follows from the corresponding property of induced bivectors.  $\square$

We conclude that a Nijenhuis tensor  $L$  on  $\mathbb{k}Q$  is enough to induce a  $\omega N$  structure on the path algebra  $\mathbb{k}\overline{Q}$ . It is important to emphasize that the Nijenhuis tensors obtained by this lifting process are quite special: for example the action of  $N(\theta)$  on an arrow  $a \in Q$  cannot involve any of the arrows in  $\overline{Q} \setminus Q$ . As we shall see in the next section, this is often a serious limitation.

## 3 Examples and applications

### 3.1 Rational Calogero-Moser system

As a first example let us consider the noncommutative  $\omega N$  manifold that underlies the phase spaces of the rational Calogero-Moser systems.

Let  $Q_\circ$  be the quiver with one vertex and one loop  $a$  and denote by  $\overline{Q}_\circ$  its double (which has an additional loop  $a^*$ ). The corresponding path algebra is the free associative algebra on the two generators  $a$  and  $a^*$ :

$$\mathbb{k}\overline{Q}_\circ = \mathbb{k}\langle a, a^* \rangle.$$

The tautological 1-form on this path algebra is  $\lambda = a^*da$  and the canonical symplectic form reads

$$\omega = da^*da. \tag{28}$$

The map  $\omega^\sharp: \text{DR}^1(\overline{Q}_\circ) \rightarrow \text{Der}(\overline{Q}_\circ)$  acts as follows: given a 1-form  $\alpha \in \text{DR}^1(\overline{Q}_\circ)$  represented by  $\sum_{c \in \overline{Q}_\circ} S_c dc$ ,

$$\omega^\sharp(\alpha) = -S_{a^*} \partial_a + S_a \partial_{a^*} \quad (29)$$

so that the Poisson bivector associated to  $\omega$  is simply

$$\pi_0 = [\partial_{a^*}, \partial_a].$$

Following [3], let us consider the endomorphism  $L: \text{Der}(Q_\circ) \rightarrow \text{Der}(Q_\circ)$  defined as follows: for every  $\theta \in \text{Der}(Q_\circ)$ ,  $L(\theta)$  is the unique derivation of  $\mathbb{k}Q_\circ$  mapping  $a$  to  $a\theta(a)$ . It is straightforward to verify that  $L$  is regular (its transpose being given by  $L^*(da) = ada$ ) and a Nijenhuis tensor on  $\mathbb{k}Q_\circ$ . The corresponding deformed tautological 1-form on  $\mathbb{k}\overline{Q}_\circ$  is

$$\lambda_L = a^* ada \quad (30)$$

with differential

$$\omega_L = d\lambda_L = da^* ada + a^* dada.$$

The complete lift of  $L$ , as defined by equation (27), is then obtained as follows. First we notice that

$$\begin{aligned} i_\theta(\omega_L) &= \theta(a^*)ada - da^* a\theta(a) + a^* \theta(a)da - a^* da \theta(a) \\ &= (\theta(a^*)a + a^* \theta(a) - \theta(a)a^*)da - a\theta(a)da^*, \end{aligned}$$

where the second equality holds in  $\text{DR}^1(\overline{Q}_\circ)$ . Then, using the expression (29) for  $\omega^\sharp$ , we get

$$N(\theta) = \omega^\sharp(i_\theta(\omega_L)) = a\theta(a)\partial_a + (\theta(a^*)a + a^* \theta(a) - \theta(a)a^*)\partial_{a^*}$$

or, more compactly,

$$N(\theta)(a, a^*) = (a\theta(a), [a^*, \theta(a)] + \theta(a^*)a). \quad (31)$$

Theorem 18 then implies that the map

$$\tilde{\pi}_1(S_a da + S_{a^*} da^*) = N(\tilde{\pi}_1(S_a da + S_{a^*} da^*)) = aS_{a^*} \partial_a + (a^* S_{a^*} - S_{a^*} a^* - S_a a) \partial_{a^*}$$

defines a double Poisson structure on  $\mathbb{k}\overline{Q}_\circ$ . Explicitly, the corresponding bivector  $\pi_1 \in \mathcal{V}^2 \overline{Q}_\circ$  reads

$$\pi_1 = [a\partial_{a^*}, \partial_a] + [a^* \partial_{a^*}, \partial_{a^*}]. \quad (32)$$

Readers of [4, 28] will recognize the previous expression as the linear Poisson bivector on  $\mathbb{k}\langle a, a^* \rangle$  induced by an appropriate associative algebra structure on  $\mathbb{k}^2$ .

Let us compute the next double Poisson structure in the hierarchy. We have

$$\begin{aligned} \tilde{\pi}_2(S_a da + S_{a^*} da^*) &= N(\tilde{\pi}_1(S_a da + S_{a^*} da^*)) \\ &= N(aS_{a^*} \partial_a + ([a^*, S_{a^*}] - S_a a) \partial_{a^*}) \\ &= a^2 S_{a^*} \partial_a + ([a^*, aS_{a^*}] + [a^*, S_{a^*}]a - S_a a^2) \partial_{a^*}, \end{aligned}$$

which corresponds to the bivector

$$\pi_2 = [a^2 \partial_{a^*}, \partial_a] + [a^* a \partial_{a^*}, \partial_{a^*}] + [a^* \partial_{a^*}, a \partial_{a^*}]. \quad (33)$$

In general, we have

$$\tilde{\pi}_m(S_a da + S_{a^*} da^*) = a^m S_{a^*} \partial_a + \left( \sum_{i=1}^m [a^*, a^{m-i} S_{a^*}] a^{i-1} - S_a a^m \right) \partial_{a^*}$$

whence

$$\pi_m = [a^m \partial_{a^*}, \partial_a] + \sum_{i=1}^m [a^* a^{m-i} \partial_{a^*}, a^{i-1} \partial_{a^*}].$$

In particular the Poisson brackets on  $\text{DR}^0(\overline{Q}_\circ)$  determined by the  $m$ -th Poisson structure read as follows:

$$\{f, g\}_m = a^m \left( \frac{\partial f}{\partial a^*} \frac{\partial g}{\partial a} - \frac{\partial g}{\partial a^*} \frac{\partial f}{\partial a} \right) + \sum_{i=1}^m a^* a^{m-i} \left( \frac{\partial f}{\partial a^*} a^{i-1} \frac{\partial g}{\partial a^*} - \frac{\partial g}{\partial a^*} a^{i-1} \frac{\partial f}{\partial a^*} \right) \quad (34)$$

where  $\frac{\partial}{\partial a}$  denotes the *necklace derivative* with respect to the arrow  $a$  [20, 15, 6].

Consider now the following family of necklace words in  $\text{DR}^0(\overline{Q}_\circ)$ :

$$I_k = \frac{1}{k} a^k \quad (k \geq 1). \quad (35)$$

As is immediate to verify, these regular functions on  $\mathbb{k}\overline{Q}_\circ$  are in involution with respect to every bracket of the hierarchy (34). Moreover, one has

$$\tilde{\pi}_1(dI_k) = \tilde{\pi}_1(a^{k-1} da) = -a^{k-1} a \partial_{a^*} = -a^k \partial_{a^*}$$

and

$$\tilde{\pi}_0(dI_{k+1}) = \tilde{\pi}_0(a^k da) = -a^k \partial_{a^*}$$

so that the functions  $I_k$  form a *Lenard chain*, that is

$$\tilde{\pi}_1(dI_k) = \tilde{\pi}_0(dI_{k+1}).$$

For later use, let us define also the following additional set of regular functions on  $\mathbb{k}\overline{Q}_\circ$ :

$$J_\ell := a^{\ell-1} a^* \quad (\ell \geq 1). \quad (36)$$

Taken together, the  $I_k$  and  $J_\ell$  span a Lie subalgebra of  $\text{DR}^0(\overline{Q}_\circ)$  with respect to each one of the brackets (34). Indeed, the following relations hold for every  $k, \ell \geq 1$  and  $m \geq 0$ :

$$\{I_k, I_\ell\}_m = 0 \quad \{J_\ell, I_k\}_m = (k + \ell + m - 2) I_{k+\ell+m-2} \quad \{J_k, J_\ell\}_m = (\ell - k) J_{k+\ell+m-2} \quad (37)$$

(with the exception that  $\{J_1, I_1\}_0 = 1$ ).

In order to relate the above constructions with the dynamics of the rational Calogero-Moser system let us descend to the space of real representations of the quiver  $\overline{Q}_\circ$  with dimension vector  $\mathbf{n} = (n)$  for some  $n \in \mathbb{N}$ . This is simply the linear space of pairs of  $n \times n$  real matrices,

$$\text{Rep}_{\mathbb{R}}(\overline{Q}_\circ, (n)) = \text{Mat}_{n \times n}(\mathbb{R}) \oplus \text{Mat}_{n \times n}(\mathbb{R}), \quad (38)$$

which can be identified with the cotangent bundle  $T^* \text{Mat}_{n \times n}(\mathbb{R})$  in the obvious way. The group  $G_{\mathbf{n}}$  defined by (5) coincides with  $\text{PGL}_n(\mathbb{R})$  and its action on the space (38) is Hamiltonian with respect to the (canonical) symplectic form  $\hat{\omega}$  on  $T^* \text{Mat}_{n \times n}(\mathbb{R})$  induced by the noncommutative symplectic form (28). As well known (see e.g. [11, Chapter 2]), the phase space of the rational  $n$ -particle Calogero-Moser system may then be recovered as a suitable symplectic quotient of the manifold  $(T^* \text{Mat}_{n \times n}(\mathbb{R}), \hat{\omega})$ , and the  $\text{PGL}_n(\mathbb{R})$ -invariant functions on  $\text{Rep}_{\mathbb{R}}(\overline{Q}_\circ, (n))$  induced by the noncommutative functions (35),

$$\hat{I}_k(X, Y) = \frac{1}{k} \text{tr } X^k,$$

give exactly the usual Calogero-Moser Hamiltonians once projected on this quotient space. (Of course, on the resulting finite-dimensional manifold only the first  $n$  of them will be functionally independent.)

This procedure is not appropriate in the present setting, however, as the quotient map obtained by the symplectic reduction process cannot be used to reduce the second Poisson structure  $\tilde{\pi}_1$  on  $\text{Rep}_{\mathbb{R}}(\overline{Q}_o, (n))$ . To do this we need to replace the ordinary symplectic reduction with a “bihamiltonian reduction”.

As described in detail in [3], a reduction of this kind is naturally viewed as a two-step process. In the first step one factors out the action of  $\text{PGL}_n(\mathbb{R})$  on  $\text{Rep}_{\mathbb{R}}(\overline{Q}_o, (n))$ , landing on a certain  $(n^2 + 1)$ -dimensional manifold  $\mathcal{P}$ . In the second step one further reduces the resulting dynamics on the  $2n$ -dimensional manifold defined by the image of the submersion  $\pi: \mathcal{P} \rightarrow \mathbb{R}^{2n}$  whose components are the invariant functions  $\hat{I}_1, \dots, \hat{I}_n$  and  $\hat{J}_1, \dots, \hat{J}_n$  (where  $\hat{J}_\ell(X, Y) = \text{tr } X^{\ell-1} Y$ ). This makes it possible to recover, for any fixed  $n$ , the phase space of the (attractive)  $n$ -particles rational Calogero-Moser system with its associated bihamiltonian structure.

**Remark 19.** We believe that a similar quotient can be constructed also starting from the space of *complex* representation of the quiver  $\overline{Q}_o$ . This generalization is needed in order to get the dynamics of the repulsive Calogero-Moser system.

**Remark 20.** Unfortunately it is not easy to write the second Poisson structure in the reduced (or “physical”) coordinates. As a matter of fact, the transformation relating the functions  $(\hat{I}_1, \dots, \hat{I}_n, \hat{J}_1, \dots, \hat{J}_n)$  to the canonical Calogero-Moser coordinates  $(q_1, \dots, q_n, p_1, \dots, p_n)$  is notoriously hard to invert. In [25] and [3] only the 2-particle case is considered; more recently the 3-particle case has been studied in [2].

**Remark 21.** In the literature concerning the construction of the Calogero-Moser phase space by symplectic reduction it is more customary to take as Hamiltonians the functions  $\hat{H}_k(X, Y) = \frac{1}{k} \text{tr } Y^k$  and  $\hat{K}_\ell(X, Y) = \text{tr } Y^{\ell-1} X$ , which are induced by the following necklace words in  $\text{DR}^0(\overline{Q}_o)$ :

$$H_k = \frac{1}{k} a^{*k} \quad \text{and} \quad K_\ell = a^{*\ell-1} a.$$

These functions also define a bihamiltonian system on  $\mathbb{k}\overline{Q}_o$  (and consequently on each representation space) if we replace the Nijenhuis tensor (31) with

$$N(\theta)(a, a^*) = ([\theta(a^*), a] + a^* \theta(a), \theta(a^*) a^*), \quad (39)$$

in which case the second Poisson structure turns out to be

$$\pi_1 = [a^* \partial_{a^*}, \partial_a] + [a \partial_a, \partial_a].$$

In this paper we stuck to the choice (31) in order to ease the comparison between our formulas and the corresponding ones in reference [3]. Notice in this respect that the Nijenhuis tensor (39) cannot be obtained by a lifting process of the kind discussed in subsection 2.4.

### 3.2 Gibbons-Hermesen system

As a second example we shall consider a noncommutative  $\omega N$  manifold related to a family of integrable systems introduced by Gibbons and Hermesen in [14]. These systems are a generalization of the rational Calogero-Moser system in which each particle has some additional degrees of freedom parametrized by a vector-covector pair living in a linear space of dimension  $r > 1$  (the case  $r = 1$  corresponds to Calogero-Moser). For the sake of notational clarity we shall



consider only the case  $r = 2$ ; however, the generalization to higher-rank cases does not present any essentially new difficulty.

Let  $Q$  be the quiver

$$\begin{array}{c} \bullet \xleftarrow{x} \bullet \\ \bullet \xrightarrow{y} \bullet \end{array} \quad (40)$$

introduced by Bielawski and Pidstrygach in [5]. On the path algebra of its double  $\overline{Q}$  we have the tautological 1-form

$$\lambda = a^* da + x^* dx + y^* dy. \quad (41)$$

The corresponding symplectic form is

$$\omega = da^* da + dx^* dx + dy^* dy.$$

The associated map  $\omega^\sharp: \text{DR}^1(\overline{Q}) \rightarrow \text{Der}(\overline{Q})$  acts on a generic 1-form  $\alpha = \sum_{c \in \overline{Q}} S_c dc$  in the following manner:

$$\omega^\sharp(\alpha) = -S_{a^*} \partial_a + S_a \partial_{a^*} - S_{x^*} \partial_x + S_x \partial_{x^*} - S_{y^*} \partial_y + S_y \partial_{y^*}. \quad (42)$$

This symplectic form will provide our first Poisson bivector on  $\mathbb{k}\overline{Q}$ ,

$$\pi_0 = [\partial_{a^*}, \partial_a] + [\partial_{x^*}, \partial_x] + [\partial_{y^*}, \partial_y].$$

Now let us consider the 1-form

$$\lambda' = a^* ada + x^* adx - y ady^*.$$

Notice that  $\lambda'$  cannot be expressed as a deformation of the 1-form (41) via a regular endomorphism of  $\text{Der}(Q)$  because of the term involving  $dy^*$ . It is, however, a rather natural extension of the 1-form (30) to the new setting.

Let us continue anyway along the same track by defining an endomorphism  $N: \text{Der}(\overline{Q}) \rightarrow \text{Der}(\overline{Q})$  using equation (27), where the role of  $\omega_L$  is now played by the 2-form

$$d\lambda' = da^* ada + a^* dada + dx^* adx + x^* dadx - dy ady^* - y dady^*.$$

By contracting with a generic derivation  $\theta$  we get

$$\begin{aligned} i_\theta(d\lambda') &= (\theta(a^*)a + [a^*, \theta(a)] - \theta(x)x^* + \theta(y^*)y) da - a\theta(a)da^* + \\ &\quad + (\theta(x^*)a + x^*\theta(a)) dx - a\theta(x)dx^* + a\theta(y^*)dy - (\theta(y)a + y\theta(a)) dy^* \end{aligned}$$

so that, using (42)

$$\begin{aligned} N(\theta) &= a\theta(a)\partial_a + (\theta(a^*)a + [a^*, \theta(a)] - \theta(x)x^* + \theta(y^*)y) \partial_{a^*} + \\ &\quad + a\theta(x)\partial_x + (\theta(x^*)a + x^*\theta(a)) \partial_{x^*} + (\theta(y)a + y\theta(a)) \partial_y + a\theta(y^*)\partial_{y^*}. \end{aligned} \quad (43)$$

Now let us consider the map  $\tilde{\pi}_1 := N \circ \tilde{\pi}_0$ . Recalling that  $\tilde{\pi}_0 = -\omega^\sharp$ , we get

$$\begin{aligned} \tilde{\pi}_1(\alpha) &= N(S_{a^*} \partial_a - S_a \partial_{a^*} + S_{x^*} \partial_x - S_x \partial_{x^*} + S_{y^*} \partial_y - S_y \partial_{y^*}) = \\ &= aS_{a^*} \partial_a + (-S_a a + [a^*, S_{a^*}] - S_{x^*} x^* - S_y y) \partial_{a^*} + \\ &\quad + aS_{x^*} \partial_x + (-S_x a + x^* S_{a^*}) \partial_{x^*} + (S_{y^*} a + y S_{a^*}) \partial_y - aS_y \partial_{y^*}. \end{aligned}$$

The corresponding bivector in  $\mathcal{V}^2\overline{Q}$  is given by

$$\pi_1 = [a\partial_{a^*}, \partial_a] + [a^*\partial_{a^*}, \partial_{a^*}] + [a\partial_{x^*}, \partial_x] + [x^*\partial_{a^*}, \partial_{x^*}] + [\partial_{y^*}, a\partial_y] + [y\partial_{a^*}, \partial_y].$$

A straightforward computation using definition 3 reveals that

$$[\pi_1, \pi_1] = 0 \quad \text{and} \quad [\pi_0, \pi_1] = 0,$$

i.e. that  $\pi_1$  is a double Poisson structure on  $\overline{Q}$  which is compatible with  $\pi_0$ . A posteriori we can conclude, using theorem 16, that the endomorphism  $N$  defined by (43) is Nijenhuis and compatible with  $\pi_0$  (in the sense that  $C_{(\pi_0, N)} = 0$ ).

Let us take as Hamiltonians the necklace words in  $\text{DR}^0(\overline{Q})$  of the following form:

$$I_k := \frac{1}{k}a^k \quad (k \geq 1) \tag{44a}$$

and

$$I_k^{(2)} := a^k(xx^* + y^*y) \quad (k \geq 0). \tag{44b}$$

It is clear that the functions  $I_k$  are linked in a Lenard chain by the two Poisson structures  $\pi_0$  and  $\pi_1$ ; the computation is essentially the same as in the Calogero-Moser case. Here, however, the additional Hamiltonians  $I_k^{(2)}$  determine another Lenard chain: in fact we have

$$dI_k^{(2)} = \sum_{i=0}^{k-1} a^i(xx^* + y^*y)a^{k-1-i}da + x^*a^kdx + a^kxdx^* + a^ky^*dy + ya^kdy^*$$

so that

$$\tilde{\pi}_1(dI_k^{(2)}) = \left(-\sum_{i=0}^{k-1} a^i(xx^* + y^*y)a^{k-i} - a^kxx^* - a^ky^*y\right)\partial_{a^*} + a^{k+1}x\partial_x - x^*a^{k+1}\partial_{x^*} + ya^{k+1}\partial_y - a^{k+1}y^*\partial_{y^*},$$

which equals

$$\tilde{\pi}_0(dI_{k+1}^{(2)}) = -\sum_{i=0}^k a^i(xx^* + y^*y)a^{k-i}\partial_{a^*} + a^{k+1}x\partial_x - x^*a^{k+1}\partial_{x^*} + ya^{k+1}\partial_y - a^{k+1}y^*\partial_{y^*}.$$

We conclude that the noncommutative functions (44) induce a bihamiltonian system on every representation space for the quiver  $\overline{Q}$ .

To explain the relationship with the Gibbons-Hernsen system let us consider the space of real representations of  $\overline{Q}$  with dimension vector  $\mathbf{n} = (n, 1)$ ,

$$\text{Rep}_{\mathbb{R}}(\overline{Q}, (n, 1)) = \text{Mat}_{n \times n}(\mathbb{R}) \oplus \text{Mat}_{n \times n}(\mathbb{R}) \oplus \text{Mat}_{n \times 1}(\mathbb{R}) \oplus \text{Mat}_{n \times 1}(\mathbb{R}) \oplus \text{Mat}_{1 \times n}(\mathbb{R}) \oplus \text{Mat}_{1 \times n}(\mathbb{R}),$$

the point corresponding to a representation  $\tau$  being given by the matrices  $(\tau_a, \tau_{a^*}, \tau_x, \tau_{y^*}, \tau_{x^*}, \tau_y)$ . As explained in [5], the phase space of the rank 2 Gibbons-Hernsen system can be obtained by identifying  $\text{Rep}_{\mathbb{R}}(\overline{Q}, (n, 1))$  with the linear space

$$V_{n,2} := \text{Mat}_{n \times n}(\mathbb{R}) \oplus \text{Mat}_{n \times n}(\mathbb{R}) \oplus \text{Mat}_{n \times 2}(\mathbb{R}) \oplus \text{Mat}_{2 \times n}(\mathbb{R})$$

consisting of quadruples  $(X, Y, v, w)$  using the bijective correspondence defined as follows:

$$X = \tau_a \quad Y = \tau_{a^*} \quad v = \begin{pmatrix} -\tau_x & \tau_{y^*} \end{pmatrix} \quad w = \begin{pmatrix} \tau_{x^*} \\ \tau_y \end{pmatrix}. \tag{45}$$

In this way the natural Hamiltonian action of the group  $G_{(n,1)} \simeq \mathrm{GL}_n(\mathbb{R})$  on the symplectic manifold  $(\mathrm{Rep}_{\mathbb{R}}(\overline{Q}, (n, 1)), \hat{\omega})$  coincides with the Hamiltonian action of  $\mathrm{GL}_n(\mathbb{R})$  on  $V_{n,2}$  used in [14] to define the phase space of the system by symplectic reduction.

The dynamics of the system is determined by taking as Hamiltonians the functions<sup>2</sup>

$$\hat{I}_k(X, Y, v, w) = \frac{1}{k} \mathrm{tr} X^k \quad \text{and} \quad \hat{H}_{k,\alpha}(X, Y, v, w) = \mathrm{tr} X^k v \alpha w, \quad (46)$$

where  $\alpha$  is any  $2 \times 2$  constant matrix (actually  $\hat{I}_k$  is just a scalar multiple of  $\hat{H}_{k,\iota}$ , where  $\iota$  is the identity matrix). These functions span a (nonabelian) Poisson algebra  $\mathcal{H}$  whose Poisson brackets are given by

$$\{\hat{H}_{k,\alpha}, \hat{H}_{\ell,\beta}\} = \hat{H}_{k+\ell, [\alpha, \beta]}.$$

The complete integrability of the system then follows from the existence of  $2n$ -dimensional abelian subalgebras of  $\mathcal{H}$ . A natural choice is to take the subalgebra spanned by the functions  $(\hat{I}_1, \dots, \hat{I}_n)$  and  $(\hat{I}_0^{(2)}, \dots, \hat{I}_{n-1}^{(2)})$  where

$$\hat{I}_k^{(2)} := \hat{H}_{k,\eta}, \quad \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Using the correspondence (45) it is immediate to check that these functions are induced, respectively, by the necklace words  $I_k$  and  $I_k^{(2)}$  in  $\mathrm{DR}^0(\overline{Q})$  defined by (44). It follows that the dynamics described by these functions on  $V_{n,2} \simeq \mathrm{Rep}_{\mathbb{R}}(\overline{Q}, (n, 1))$  is bihamiltonian with respect to the induced Poisson structures  $\tilde{\pi}_0$  and  $\tilde{\pi}_1$ .

Unfortunately this is not enough to conclude that the induced dynamics on the quotient manifold is also bihamiltonian. In fact we are faced with exactly the same problem that arose in the Calogero-Moser case: in order to reduce the second Poisson structure we cannot use the projection map coming from the symplectic reduction *à la* Gibbons-Hermsen, as this map will not preserve  $\pi_1$ . Instead we have to devise an appropriate bihamiltonian reduction scheme similar to the one set up in [3] for the case  $r = 1$ . Namely, we should reduce the bihamiltonian manifold consisting of the linear space  $\mathrm{Rep}_{\mathbb{R}}(\overline{Q}, (n, 1))$  equipped with the two compatible Poisson structures  $\tilde{\pi}_0$  and  $\tilde{\pi}_1$  to a suitable  $4n$ -dimensional bihamiltonian manifold, which then must be identified with the usual phase space of the rank 2 Gibbons-Hermsen system.

This is a non-trivial problem that we are not going to tackle here. However, let us briefly sketch a possible way to construct such a quotient. Following [3] it is natural to look for a two-step projection,

$$\mathrm{Rep}_{\mathbb{R}}(\overline{Q}, (n, 1)) \longrightarrow \mathcal{P} \longrightarrow \mathbb{R}^{4n},$$

where the first step involves the definition of a  $(n^2 + 4n)$ -dimensional slice  $\mathcal{P}$  for the action of  $\mathrm{GL}_n(\mathbb{R})$  on the  $(2n^2 + 4n)$ -dimensional space  $\mathrm{Rep}_{\mathbb{R}}(\overline{Q}, (n, 1))$ . Once this slice has been defined, the second projection may again be performed by the following procedure:

- 1) one selects a set of  $4n$  regular  $\mathrm{GL}_n(\mathbb{R})$ -invariant functions on  $\mathrm{Rep}_{\mathbb{R}}(\overline{Q}, (n, 1))$  which span a Poisson subalgebra with respect to both brackets and whose Jacobian matrix with respect to the reduced Gibbons-Hermsen coordinates is nondegenerate;
- 2) one takes the submersion  $\mathcal{P} \rightarrow \mathbb{R}^{4n}$  whose components are given by those functions.

Such a set of generators may consist, for example, of the  $2n$  Hamiltonians  $\hat{I}_k, \hat{I}_k^{(2)}$  considered before supplemented with the  $n$  functions

$$\hat{J}_\ell := \mathrm{tr} X^{\ell-1} Y \quad (1 \leq \ell \leq n),$$

---

<sup>2</sup>As in the previous subsection we modify the usual Hamiltonians by exchanging the matrices  $X$  and  $Y$ .

familiar from the Calogero-Moser case, and with the further  $n$  functions  $\hat{J}_0^{(2)}, \dots, \hat{J}_{n-1}^{(2)}$ , where

$$\hat{J}_\ell^{(2)} := \hat{H}_{\ell, e_{12}}, \quad e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which are needed to recover the additional degrees of freedom contained in the matrices  $v$  and  $w$ .

## 4 Final remarks

We believe that the formalism presented in this paper may be of help in finding bihamiltonian structures for many other classical finite-dimensional integrable systems. The most obvious case to be considered next is that of Calogero-Moser systems with trigonometric/hyperbolic potentials and their generalizations with internal degrees of freedom (see for instance the survey [27]). In this connection let us observe that, as pointed out by Bielawski [4, Remark 7.3], the Poisson bivector  $\pi_1$  given in eq. (32) also induces, on a suitable open subset of  $\text{Rep}_{\mathbb{K}}(\overline{Q}_o, n)$ , the symplectic structure of the trigonometric Calogero-Moser system. The compatibility between  $\pi_1$  and the Poisson bivector  $\pi_2$  given in eq. (33) then suggests that the bihamiltonian description of this system hinted at in [1] may be derived from this pair of double Poisson structures.

Another promising source of examples may come from the very general class of integrable systems arising from the Coulomb branch of the moduli space of vacua in four-dimensional  $N = 2$  supersymmetric gauge theories [31, 32, 9]. As the referee pointed out to us, many explicit examples of systems of this kind have been derived, most recently by Dorey and Zhao [10], starting from elliptic quiver gauge theories. Interpreting these systems from a noncommutative-geometric point of view seems to be an interesting problem.

The last issue we would like to mention is related to the notion of *duality* between integrable systems introduced by Ruijsenaars in [30] and later reinterpreted in terms of the symplectic reduction of two families of commuting Hamiltonians on a higher-dimensional symplectic manifold [17, 13]. Being a relation between canonical coordinates on actual phase spaces, the Ruijsenaars duality transformation can be implemented only at the level of symplectic quotients, and is thus invisible at the noncommutative level. However, in many cases the data to be provided as input for the construction (namely the “big” phase space, the symplectic form and the two families of commuting Hamiltonians) can be interpreted in terms of geometric objects on quiver representation spaces.

A relevant example is provided by the well known duality between the trigonometric Calogero-Moser(-Sutherland) system and the rational Ruijsenaars-Schneider system, which was put on a firm geometric basis by Fehér and Klimčík [12]. The input data for their construction seem to admit a noncommutative-geometric interpretation. If so, it should be possible to derive a bihamiltonian structure for both systems starting from the same noncommutative PN structure pointed out above (see again the related computations in [1]).

## Acknowledgments

This work was partially supported by the PRIN “Geometria delle varietà algebriche” and by the University of Genoa’s research grant “Aspetti matematici della teoria dei campi interagenti e quantizzazione per deformazione”. The authors are grateful to the referee, whose suggestions have been incorporated in section 4. A.T. would like to thank the Department of Mathematics at the University of Genoa for the kind hospitality during the period in which this paper was written.

## References

- [1] I. ANICETO, J. AVAN, AND A. JEVICKI, *Poisson structures of Calogero–Moser and Ruijsenaars–Schneider models*, Journal of Physics A: Mathematical and Theoretical, 43 (2010), pp. 185–201. [arXiv:0912.3468](#).
- [2] J. AVAN AND E. RAGOUCY, *Rational Calogero–Moser model: Explicit form and  $r$ -matrix of the second Poisson structure*, SIGMA, 8 (2012), p. 079. [arXiv:1207.5368](#).
- [3] C. BARTOCCI, G. FALQUI, I. MENCATTINI, G. ORTENZI, AND M. PEDRONI, *On the geometric origin of the bi-Hamiltonian structure of the Calogero–Moser system*, Int. Math. Res. Not., 2010 (2010), pp. 279–296. [arXiv:0902.0953](#).
- [4] R. BIELAWSKI, *Quivers and Poisson structures*, Manuscripta Math., 141 (2013), pp. 29–49. [arXiv:1108.3222](#).
- [5] R. BIELAWSKI AND V. PIDSTRYGACH, *On the symplectic structure of instanton moduli spaces*, Adv. in Math., 226 (2011), pp. 2796–2824. [arXiv:0812.4918](#).
- [6] R. BOCKLANDT AND L. LE BRUYN, *Necklace Lie algebras and noncommutative symplectic geometry*, Math. Z., 240 (2002), pp. 141–167. [arXiv:math/0010030](#).
- [7] M. BRION, *Representations of quivers*, in Geometric methods in representation theory. I, vol. 24 of Sémin. Congr., Soc. Math. France, Paris, 2012, pp. 103–144.
- [8] J. CUNTZ AND D. QUILLEN, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc., 8 (1995), pp. 251–289.
- [9] R. DONAGI AND E. WITTEN, *Supersymmetric Yang–Mills theory and integrable systems*, Nuclear Phys. B, 460 (1996), pp. 299–334. [arXiv:hep-th/9510101](#).
- [10] N. DOREY AND P. ZHAO, *Solution of quantum integrable systems from quiver gauge theories*. [arXiv:1512.09367](#).
- [11] P. ETINGOF, *Calogero–Moser systems and representation theory*, Zurich Lectures in Advanced Mathematics, EMS, Zurich, 2007. [arXiv:math/0606233](#).
- [12] L. FEHÉR AND C. KLIMČÍK, *On the duality between the hyperbolic Sutherland and the rational Ruijsenaars–Schneider models*, J. Phys. A, 42 (2009), pp. 185202, 13. [arXiv:0901.1983](#).
- [13] V. FOCK, A. GORSKY, N. NEKRASOV, AND V. RUBTSOV, *Duality in integrable systems and gauge theories*, J. High Energy Phys., (2000), pp. Paper 28, 40. [arXiv:hep-th/9906235](#).
- [14] J. GIBBONS AND T. HERMSEN, *A generalization of the Calogero–Moser system*, Physica, 11D (1984), pp. 337–348.
- [15] V. GINZBURG, *Non-commutative symplectic geometry, quiver varieties, and operads*, Math. Res. Lett., 8 (2001), pp. 377–400. [arXiv:math/0005165](#).
- [16] ———, *Lectures on noncommutative geometry*. [arXiv:math/0506603](#), 2005.
- [17] A. GORSKY AND N. NEKRASOV, *Relativistic Calogero–Moser model as gauged WZW theory*, Nuclear Physics B, 436 (1995), pp. 582–608. [arXiv:hep-th/9401017](#).
- [18] A. IBORT, F. MAGRI, AND G. MARMO, *Bihamiltonian structures and Stäckel separability*, J. Geom. Phys., 33 (2000), pp. 210–228.

- [19] M. KAROUBI, *Homologie cyclique et K-théorie*, Astérisque, (1987), p. 147.
- [20] M. KONTSEVICH, *Formal (non)commutative symplectic geometry*, in The Gel'fand Mathematical Seminars, 1990–1992, Birkhäuser Boston, Boston, MA, 1993, pp. 173–187.
- [21] Y. KOSMANN-SCHWARZBACH AND F. MAGRI, *Poisson-Nijenhuis structures*, Annales de l'I.H.P., 53 (1990), pp. 35–81.
- [22] C. I. LAZAROIU, *On the non-commutative geometry of topological D-branes*, Journal of High Energy Physics, 2005 (2005), p. 032. [arXiv:hep-th/0507222](#).
- [23] J.-L. LODAY, *Cyclic homology*, vol. 301 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin, second ed., 1998.
- [24] F. MAGRI, *A simple model of the integrable Hamiltonian equation*, J. Math. Phys., 19 (1978), pp. 1156–1162.
- [25] F. MAGRI AND T. MARSICO, *Some developments of the concept of Poisson manifold in the sense of A. Lichnerowicz*, in Gravitation, Electromagnetism, and Geometric Structures, G. Ferrarese, ed., Pitagora editrice, Bologna, 1996, pp. 207–222.
- [26] F. MAGRI AND C. MOROSI, *A geometrical characterization of Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds*, Quaderno S 19/1984, Università degli studi di Milano, 1984.
- [27] N. NEKRASOV, *Infinite-dimensional algebras, many-body systems and gauge theories*, in Moscow Seminar in Mathematical Physics, vol. 191 of Amer. Math. Soc. Transl. Ser. 2, Amer. Math. Soc., Providence, RI, 1999, pp. 263–299.
- [28] A. ODESSKII, V. RUBTSOV, AND V. SOKOLOV, *Double Poisson brackets on free associative algebras*, in Noncommutative birational geometry, representations and combinatorics, vol. 592 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2013, pp. 225–239. [arXiv:1208.2935](#).
- [29] A. PICHEREAU AND G. VAN DE WEYER, *Double Poisson cohomology of path algebras of quivers*, J. Algebra, 319 (2008), pp. 2166–2208. [arXiv:math/0701837](#).
- [30] S. N. M. RUIJSENAARS, *Action-angle maps and scattering theory for some finite-dimensional integrable systems. I. The pure soliton case*, Comm. Math. Phys., 115 (1988), pp. 127–165.
- [31] N. SEIBERG AND E. WITTEN, *Electric-magnetic duality, monopole condensation, and confinement in  $N = 2$  supersymmetric Yang-Mills theory*, Nuclear Phys. B, 426 (1994), pp. 19–52.
- [32] ———, *Monopoles, duality and chiral symmetry breaking in  $N = 2$  supersymmetric QCD*, Nuclear Phys. B, 431 (1994), pp. 484–550.
- [33] A. TACCHHELLA, *An introduction to associative geometry with applications to integrable systems*, Journal of Geometry and Physics (to appear). [arXiv:1611.00644](#).
- [34] F.-J. TURIÉL, *Structures bihamiltoniennes sur le fibré cotangent*, C. R. Acad. Sci. Paris Sér. I Math., 315 (1992), pp. 1085–1088.

- [35] M. VAN DEN BERGH, *Double poisson algebras*, Trans. Amer. Math. Soc., 360 (2008), pp. 5711–5769. [arXiv:math/0410528](#).

DIPARTIMENTO DI MATEMATICA, UNIVERSITÀ DI GENOVA, VIA DODECANESO 35, 16146  
GENOVA, ITALY

*Email addresses:* [bartocci@dima.unige.it](mailto:bartocci@dima.unige.it), [altacch@gmail.com](mailto:altacch@gmail.com)